ENM 5220 Numerical Solution of PDEs | Final Project Proposal

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1. Project Title: ADI Time-Stepping for a 2D Reaction–Diffusion System on a Fixed Domain

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2. Source Material

A. Madzvamuse and P. K. Maini, "Velocity-induced numerical solutions of reaction-diffusion systems on continuously growing domains," *Journal of Computational Physics*, vol. 225, no. 1, pp. 100–119, 2007. https://doi.org/10.1016/j.jcp.2006.11.022

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3. Summary of Source Work

Madzvamuse and Maini (2007) look at how mesh structure affects numerical solutions of a coupled nonlinear reaction-diffusion system on fixed and growing 2D domains. The underlying physics is Turing's diffusion-driven instability: the counterintuitive result that diffusion, usually a stabilizing influence, can actually destabilize a uniform steady state and drive the formation of spatial patterns. The paper uses the Schnakenberg model as its test case: $f(u,v) = a - u + u^2v$ and $g(u,v) = b - u^2v$.

Two schemes are compared throughout. The first is an ADI finite difference method, applied after a Lagrangian coordinate transformation maps the problem onto a fixed grid. The ADI scheme splits each timestep into two half-steps, alternating which spatial direction is treated implicitly, which keeps the linear systems tridiagonal and cheap to solve. The second is a moving grid finite element method with a semi-implicit second-order BDF time integrator (MGFEM-2SBDF) on an unstructured triangular mesh.

Section 4 looks at the fixed-domain case specifically. With the growth function set to $\rho(t) = 1$, the equations reduce to the standard Schnakenberg system on a unit square, and the paper runs experiments under both zero-flux and periodic boundary conditions for $\gamma = 114$, 1000, and 5000. Both schemes produce qualitatively similar patterns, which validates the ADI approach for fixed domains. One interesting observation is that the ADI scheme on a uniform square mesh tends to select more symmetric patterns than the finite element scheme, which is a consequence of the mesh geometry implicitly biasing the solution.

4.Project Description

The goal of this project is to reproduce the Section 4 fixed-domain results using an ADI solver written in MATLAB. The equations being solved are the Schnakenberg system on the unit square:

1. $u_t = \nabla^2 u + \gamma(a - u + u^2v)$
2. $v_t = d\nabla^2 v + \gamma(b - u^2v)$

with zero-flux boundary conditions and initial conditions taken as small random perturbations around the homogeneous steady state. Parameters match the paper: $a = 0.1$, $b = 0.9$, $d = 10$, with γ varied over 114, 1000, and 5000.

The scheme is Peaceman-Rachford ADI applied to the coupled system. At each timestep, the diffusion operator is split into x and y directional implicit sweeps. The nonlinear reaction terms are handled explicitly, evaluated at the previous timestep and folded into the right-hand side, which avoids nonlinear solves at each step and keeps things tractable. The specific tasks are:

1. Deriving the ADI discretization by hand for both u and v, working out the Neumann boundary condition treatment at each half-step carefully. This is where most of the subtle bugs tend to live.
2. Implementing the scheme in MATLAB on a uniform grid with $\Delta x = \Delta y = 1/150$, using the Thomas algorithm for the tridiagonal solves.
3. Running simulations to $t_F = 5$ with $\Delta t = 10^{-2}$ for all three γ values and plotting steady-state solutions for comparison against Figures 1 and 2 of the paper.
4. Checking convergence on a pure diffusion test case with a known solution, and doing a short mesh sensitivity study to probe the pattern selection behavior the paper discusses.

The trickiest part will be getting the Neumann boundary conditions right at each ADI half-step for a coupled system. In the scalar case this is routine, but with two coupled species and semi-implicit reaction terms there are a few ways to get it subtly wrong, and the effect usually shows up as a drift in the solution near the boundary or a slow degradation of the convergence rate. The expected outcome is to accurately reproduce the pattern formation results reported in Section 4 of the reference paper and verify convergence and stability properties of the ADI scheme.

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5. Project Plan

Week	Tasks
1 (Apr 1-7)	Read Section 4 carefully; work out ADI splitting and BC treatment by hand; set up grid in MATLAB
2 (Apr 8-14)	Build the ADI solver; test on pure diffusion and verify convergence rate
3 (Apr 15-21)	Add reaction terms; run the three γ cases; compare figures against the paper
4 (Apr 22-28)	Mesh sensitivity study; stability checks; start writing the report
Final Week	Clean up code and finish the report